Computing π

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Roots of polynomial and trigonometric functions

If a and b are non-zero real numbers then they are roots of the equation

$$(1-\frac{x}{a})(1-\frac{x}{b})=0$$

which is the same as the equation

$$1-(\frac{1}{a}+\frac{1}{b})x+\frac{1}{ab}x^2=0$$

Thus the negative of the coefficient of x is the sum of the reciprocal of the roots. Replace x by x^2 and a, b by a^2, b^2:

$$(1-\frac{x^2}{a^2})(1-\frac{x^2}{b^2})=1-(\frac{1}{a^2}+\frac{1}{b^2})x^2+\frac{1}{a^2b^2}x^4=0$$

Now the roots are $\pm a$, $\pm b$. Thus the coefficient of x^2 is the sum of the squares of the reciprocals of the positive roots.

polynomials of degree 2

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Clearly, if we pick any three non-zero real numbers a, b, c then $\pm, \pm b, \pm c$ are the roots of the equation

$$(1-\frac{x^2}{a^2})(1-\frac{x^2}{b^2})(1-\frac{x^2}{c^2})=0$$

or the equation

$$1-(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})x^2+(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2})x^4-\frac{1}{a^2b^2c^2}x^6=0.$$

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induction!

What if we pick n non-zero real numbers a_i, $i = 1, 2, \ldots, n$. Well, we can find the polynomial equation which has $\pm a_i$, $i = 1, 2, \ldots, n$ as its roots, namely

$$\prod_{i=1}^{n}(1 - \frac{x^2}{a_i^2}) = 0.$$

One final time, we rewrite this equation as

$$1 - (\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2})x^2 + \cdots = 0$$

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Roots of trigonometric equations?

What about the equation

$$\sin \mathrm{x} = \mathrm{x} - \frac{\mathrm{x}^3}{3!} + \frac{\mathrm{x}^5}{5!} - \frac{\mathrm{x}^7}{7!} + \cdots = 0??$$

We know that the roots of this equation $\sin \mathrm{x} = 0$ are

$$0, \pm\pi, \pm 2\pi, \ldots .$$

To ensure that the roots are non-zero, we consider the equation

$$\frac{\sin \mathrm{x}}{\mathrm{x}} = 1 - \frac{\mathrm{x}^2}{3!} + \frac{\mathrm{x}^4}{5!} - \frac{\mathrm{x}^6}{7!} + \cdots = 0$$

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Wishful thinking?

We have pointed out at the outset that if p is a polynomial with roots $\pm a_i$, $i = 1, 2, \ldots, n$ then it has the factorization: $\prod_{i=1}^{n}(1 - \frac{x^2}{a_i^2})$. The trigonometric function $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$ is sort of like a polynomial of infinite degree – consequently, has infinitely many zeros! Thus it is natural to expect a factorization of the form:

$$\frac{\sin x}{x} = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots .$$

Do you see any problem in concluding that the coefficient of x^2 in $\frac{\sin x}{x}$ is equal to $\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots$?

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The amazing formula of Euler

If $\frac{\sin x}{x}$ were to admit a factorization involving its roots, then it would be natural to expect that sum of the squares of the reciprocal of these roots equal the coefficient of x^2 as in the case of ordinary polynomials, that is, we would have

$$\frac{\pi^2}{3!} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots \frac{1}{n^2} + \cdots .$$

This is the formula for computing π which was discovered by Euler in 1735.

A proof of the Euler formula

A trigonometric function

Recall that $2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi)$. We then see that

$$\begin{aligned}
\frac{1}{2} 2\sin\frac{1}{2}\mathrm{x} &= \sin\frac{1}{2}\mathrm{x} \\
\cos\mathrm{x} 2\sin\frac{1}{2}\mathrm{x} &= \sin\frac{3}{2}\mathrm{x} - \sin\frac{1}{2}\mathrm{x} \\
\vdots &= \vdots \\
\cos\mathrm{nx} 2\sin\frac{1}{2}\mathrm{x} &= \sin\frac{2\mathrm{n}+1}{2}\mathrm{x} - \sin\frac{2\mathrm{n}-1}{2}\mathrm{x}.
\end{aligned}$$

Adding these, we obtain:

$$2\sin\frac{1}{2}\mathrm{x}\,(\frac{1}{2} + \cos\mathrm{x} + \cos 2\mathrm{x} + \cdots + \cos\mathrm{nx}) = \sin\frac{2\mathrm{n}+1}{2}\mathrm{x}$$

For each $n \in \mathbb{N}$, let us define a the function f_n by the formula

$$f_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin \frac{2n+1}{2} x}{2 \sin \frac{x}{2}}.$$

Since

$$\begin{aligned} \int_0^{k\pi} u \cos u \, du &= -u \sin u \Big|_0^{k\pi} + \int_0^{k\pi} \sin u \, du \\ &= \cos u \Big|_0^{k\pi} \\ &= (-1)^k - 1, \end{aligned}$$

it follows that $\int_0^{\pi} x \cos x \, dx = \frac{1}{k^2} \int_0^{k\pi} u \cos u \, du$, where $kx = u$.

Hence

$$\begin{aligned} \mathrm{E_n} : &= \int_0^\pi \mathrm{x f_n(x)}\, \mathrm{dx} \\ &= \int_0^\pi (\frac{1}{2} + \cos \mathrm{x} + \cos 2\mathrm{x} + \cdots + \cos \mathrm{nx})\, \mathrm{dx} \\ &= \frac{\pi^2}{4} + \sum_{\mathrm{k=1}}^{\mathrm{n}} (\frac{(-1)^{\mathrm{k}} - 1}{\mathrm{k}^2}). \end{aligned}$$

Since the even terms on the right hand side are zero, we have

$$\frac{1}{2}\mathrm{E_{2n-1}} = \frac{\pi^2}{8} - \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{(2\mathrm{k}-1)^2}$$

The Goal

Our main goal now is to prove that $\lim_{n\to\infty} E_{2n-1} = 0$. As a consequence, we have the following:

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Hence we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

It then follows that

$$\frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Let g be the function defined by the formula

$$\mathrm{g(x)} = \frac{\mathrm{d}}{\mathrm{dx}}\Big(\frac{\frac{\mathrm{x}}{2}}{\sin\frac{\mathrm{x}}{2}}\Big),\ 0 \le \mathrm{x} \le \frac{\pi}{2}.$$

Integrating by parts, we have:

$$\begin{aligned}
\mathrm{E_{2n-1}} &= \int_0^{\frac{\pi}{2}} \frac{\frac{\mathrm{x}}{2}}{\sin\frac{\mathrm{x}}{2}} \sin(4\mathrm{n}-1)\frac{\mathrm{x}}{2}\,\mathrm{dx} \\
&= \tfrac{2}{4\mathrm{n}-1}\ \Big(\frac{\frac{\mathrm{x}}{2}}{\sin\frac{\mathrm{x}}{2}} \cos\tfrac{4\mathrm{n}-1}{2}\mathrm{x}\Big|_0^{\pi} + \int_0^{\pi} \mathrm{g(x)} \cos\tfrac{4\mathrm{n}-1}{2}\mathrm{x}\,\mathrm{dx}\Big) \\
&= \Big(\frac{2}{4\mathrm{n}-1}\Big)\Big(1 + \int_0^{\pi} \mathrm{g(x)} \cos\frac{4\mathrm{n}-1}{2}\mathrm{x}\,\mathrm{dx}\Big)
\end{aligned}$$

We see that g is an increasing function on the interval $[0, \pi]$. So, it is bounded by $g(\pi) = \frac{1}{2}$. It now follows that $E_{2n-1} \to 0$ as $n \to \infty$ since

$$\int_0^{\pi} g(x) \cos \frac{4n-1}{2} x \, dx \leq g(\pi) \int_0^{\pi} \cos \frac{4n-1}{2} \, dx \to 0$$

as $n \to \infty$.

Gauss Circle Problem

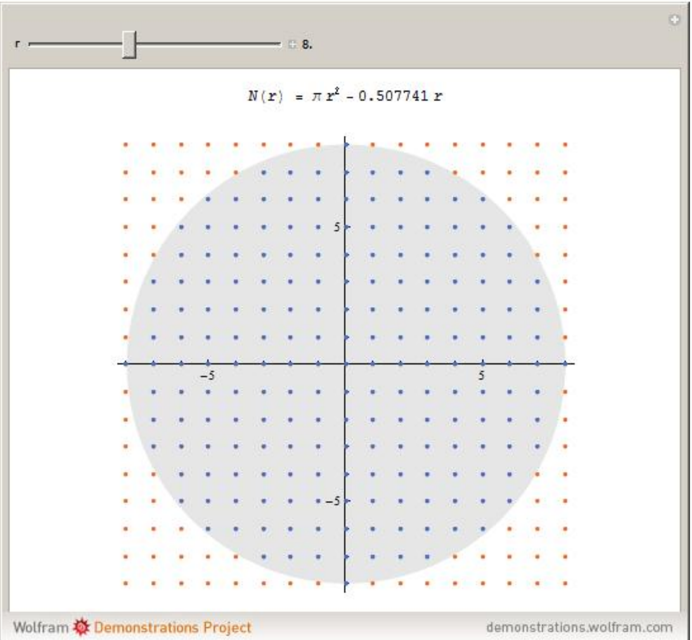
Lattice points in a circle

How many lattice points lie on or inside the circle centered at the origin and with radius r? Let L(r) be the number of such points. If we gather a bit of data, we see that L(r) grows quadratically with respect to r which leads to consideration of the function $\frac{L(r)}{r^2}$. Here is some data:

$$\begin{aligned} \frac{L(10)}{10^2} &= 3.17 \\ \frac{L(100)}{100^2} &= 3.1417 \\ \frac{L(1000)}{1000^2} &= 3.141549 \end{aligned}$$

The result must be something like $L(r) \sim \pi r^2$ as $r \to \infty$.

A picture



r
8.
N(r) = π r² - 0.507741 r
5
−5
−5
5
Wolfram Demonstrations Project
demonstrations.wolfram.com

Some justification

Of course, πr^2 is the area of the region D_r bounded by the circle $x^2 + y^2 = r^2$. It stands to reason that the area of the circle is a good approximation to the number of lattice points it contains.

Each lattice point $P = (x, y)$ contributing to $L(r)$ may be thought of as determining the square $[x, x+1] \times [y, y+1]$.Thus $L(r)$ can be thought of as the total area of the squares whose lower left corner is contained in the region D_r. Indeed for $r \geq 8$, it turns out that

$$|L(r) - \pi r^2| \leq 10r.$$

Let $P = (x, y)$ be a lattice point inside D_r and S_P be the uniquely associated square to P. Since the area of the unit square is $\sqrt{2}$, it follows that S_P lies in the region $D_{r+\sqrt{2}}$. Thus

$$L(r) \le \pi(r + \sqrt{2})^2 = \pi r^2 + 2\pi\sqrt{2}r + 2\pi.$$

Similarly, suppose $P = (x, y)$ is a lattice point with $\sqrt{x^2 + y^2} \le r - \sqrt{2}$. Then the entire unit square S_P lies in the region $D_{r-\sqrt{2}}$. This gives the estimate

$$L(r) \ge \pi(r - \sqrt{2})^2 = \pi r^2 - 2\pi\sqrt{2}r + 2\pi.$$

Thus

$$|L(r) - \pi r^2| \le 2\pi + 2\sqrt{2}\pi r \le 7 + 9r < 10r,$$

where the last inequality holds for $r \ge 8$.